International Journal of Theoretical Physics, Vol. 38, No. 10, 1999

Classification of Nilpotent Lie Superalgebras of Dimension Five. II

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Received April 4, 1999

In this paper a classification is made for all nilpotent Lie superalgebras (graded Lie algebras) of maximum dimension five. The superversion of the Kirillov lemma for nilpotent Lie superalgebra is given with its application to this classification.

1. INTRODUCTION

A Lie superalgebra $L = L_0 \oplus L_1$ is a superalgebra over a base field $K = R$ or C with an operation $[.\,,.]$ satisfying the following axioms:

(i) $[x_\alpha, x_\beta] = -(-1)^{\alpha,\beta}\,[x_\beta, x_\alpha]$.

(ii) $(-1)^{\alpha,\gamma}[[x_\alpha, x_\beta], x_\gamma] + (-1)^{\alpha,\beta}\,[[x_\beta, x_\gamma], x_\alpha] + (-1)^{\gamma,\beta}[[x_\gamma, x_\alpha], x_\beta] = 0$, $x_\alpha \in L_\alpha$, $x_\beta \in L_\beta$; $\alpha, \beta \in \{0, 1\} = Z_2$.

L_0 is called the even part, and is a Lie algebra, and L_1 is called the odd part, and is an L_0-module by restriction of the adjoint representation [1]. We say that $L = L_0 \oplus L_1$ and $L' = L'_0 \oplus L'_1$ are equivalent if there are isomorphisms $L_0 \to L'_0$ and $L_1 \to L'_1$ which preserve the bracket multiplication. We say also that L is trivial if $[L_1, L_1] = \{0\}$; otherwise L is nontrivial. It is also worth noting that the structure constants of a trivial Lie superalgebra, L say, can be interpreted as the structure constants of an associated Lie algebra, L^* say, provided that we replace the zero anticommutator of L by the zero commutator of L^*. However, under this correspondence, inequivalent Lie superalgebras can lead to equivalent Lie algebras. Other departures from ordinary Lie theory include the fact that Lie's theorem is not valid, that

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0020-7748/99/1000-268 1$16.00/0 © 1999 Plenum Publishing Corporation

Cartan's criterion for simplicity only works in one direction, that there is no obvious analog of Levi's theorem, and that there can exist zero divisors in the enveloping algebra [1, 5].

In the present work we give a classification of nilpotent Lie superalgebras, which are not Lie algebras, up to dimension five.

The classification of Lie superalgebras of dimension four does not encounter any simple Lie superalgebras (the smallest simple Lie superalgebra is of dimension five, the so-called di-spin algebra [5]) and all of these are solvable [6–8, 11].

Let $L = L_0 \oplus L_1$ be a Lie superalgebra, define a sequence of ideals of L by $L^{(0)} = L$, $L^{(1)} = [L, L]$, $L^{(2)} = [L, L^{(1)}], \ldots, L^{(i)} = [L, L^{(i-1)}]$. Then a Lie superalgebra L is called nilpotent if there exists i and $L^{(i)} = (0)$; i is called the degree of nilpotency. An ideal I of L is a superideal if $\sigma(I) = I$, where σ is an automorphism of L defined by $\sigma(x_0 + x_1) = (x_0 - x_1)$ for $x_0 \in L_0$, $x_1 \in L_1$.

For a Lie superalgebra L, it is known that L is nilpotent if and only if ad_x is a nilpotent operator for all $x \in L$, where ad_x is the adjoint representation of L [1, 9]; it is also known that Engel's theorem is valid, but Lie's theorem is not valid for nilpotent Lie superalgebra [1, 5]. Now we study the nilpotent Lie superalgebra over $K = R$ or C with dimension ≤ 5 and $[L, L] \neq 0$; otherwise all results are trivial.

We tabulate the families of equivalence classes of the indecomposable nilpotent Lie superalgebra of maximum dimension five (Table I). We say that $L = L_0 \oplus L_1$ is an (m, n) algebra if dim L_0 (resp. L_1) is m (resp. n). For the labeling of algebras, the letters A, B, C, D, E with integral superscript i denote equivalence classes of algebras of dimension d, $d = 1$ for A, $d = 2$ for B, $d = 3$ for C, $d = 4$ for D, and $d = 5$ for E. The symbol L is the associated Lie algebra when we take its structure constants as the structure constants of the trivial Lie superalgebra and the superscript i is omitted whenever its range is just the integer one [6–8]. $L^i_{(m,n)}$ represents the families of the equivalence classes of the indecomposable nilpotent Lie superalgebra of maximum dimension $(m + n) \leq 5$.

Proposition 1.1. Let $L = L_0 + L_1$ be a Lie superalgebra of type $(1, n)$. Then:

(i) Either $[L_0, L_1] = (0)$ or $[L_1, L_1] = (0)$.

(ii) L is nilpotent if and only if there exists a basis $B_L = B_{L_0} \cup B_{L_1} = \{x_0\} \cup \{x_1^1, x_1^2, \ldots, x_1^n\}$ such that either $[x_1^i, x_1^i] = a_i x_0$, $a_i \in \{-1, 0, 1]$ for $K = R$, $a_i \in \{0, 1\}$ for $K = C$, $i = 1, 2, \ldots, n$, and all other Lie products are zero or the matrix representation of ad_{x_0} has the Jordan form,

Table I.

Type	L	Characterization	Relation	Comments	K
(1, 0)	—	—	—	—	—
(0, 1)	A	$L = \langle 0\rangle \oplus \langle x_1\rangle$, $[x_1, x_1] = 0$	Trivial	Equivalent to $A_{1,1}$ and abelian	R, C
(1, 1)	$(A_{1,1} + A)$	$L = \langle x_0\rangle \oplus \langle x_1\rangle$, $[x_1, x_1] = x_0$	$L^1_{(1,1)}$	Nontrivial	R, C
(2, 0)	—	$L = \langle x_0, y_0\rangle \oplus \langle 0\rangle$	—	Abelian Lie algebra	R, C
(0, 2)	—	—	—	—	—
(3, 0)	—	$L = \langle x_0, y_0, z_0 \oplus \langle 0\rangle, [x_0, y_0] = z_0$	$L^2_{(3,0)}$	Heisenberg Lie algebra	R, C
(2, 1)	$(2A_{1,1} + A)$	$L = \langle x_0, y_0\rangle \oplus \langle x_1\rangle, [x_1, x_1] = x_0, [x_0, y_0] = 0$	Derived from $L^1_{(1,1)}$	Nontrivial	R, C
(1, 2)	C	$L = \langle x_0\rangle \oplus \langle x_1, y_1\rangle, [x_0, x_1] = y_1$	$L^3_{(1,2)}$	Trivial	R, C
(1, 2)	$(A_{1,1} + 2A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0$	$L^4_{(1,2)}$	Nontrivial	R, C
(1, 2)	$(A_{1,1} + 2A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = x_0$	$L^5_{(1,2)}$	Nontrivial	R
(0, 3)	—	—	—	—	—
(4, 0)	—	$L = \langle x_0, y_0, z_0, v_0\rangle, [x_0, y_0] = z_0, [x_0, z_0] = v_0$	$L^6_{(4,0)}$	Lie algebra	R, C
(3, 1)	$(A_{3,1} + A)$	$L = \langle x_0, y_0, z_0\rangle \oplus \langle x_1\rangle, [x_0, y_0] = z_0, [x_1, x_1] = z_0$	$L^7_{(3,1)}$	Nontrivial Heisenberg Lie superalgebra	R, C
(2, 2)	$(C + A)$	$L = \langle x_0, y_0\rangle \oplus \langle x_1, y_1\rangle, [x_0, x_1] = y_1, [x_1, x_1] = y_0$	$L^8_{(2,2)}$	Nontrivial	R, C
(2, 2)	$(2A_{1,1} + 2A)^1$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0,]x_1, y_1] = 1/2(x_0 + y_0)$	$L^9_{(2,2)}$	Nontrivial	R, C
(2, 2)	$(2A_{1,1} + 2A)^2$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0$	$L^{10}_{(2,2)}$	Nontrivial	R, C
(2, 2)	$(2A_{1,1} + 2A)^3$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0, [x_1, y_1] = (x_0 - y_0)$	$L^{11}_{(2,2)}$	Nontrivial	R, C
(2, 2)	$(2A_{1,1} + 2A)^4$	$[x_1, x_1] = x_0, [y_1, y_1] = y_0, [x_1, y_1] = x_0$	$L^{12}_{(2,2)}$	Nontrivial	R, C

Table I. Continued

Type	L	Characterization	Relation	Comments	K
(1, 3)	D	$L = \langle x_0 \rangle \oplus \langle x_1, y_1, z_1 \rangle$, $[x_0, x_1] = y_1$, $[x_0, y_1] = z_1$	$L^{13}_{(1,3)}$	Trivial	R, C
(1, 3)	$(A_{1,1} + 3A)^1$	$[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[z_1, z_1] = x_0$	$L^{14}_{(1,3)}$	Nontrivial	R, C
(1, 3)	$(A_{1,1} + 3A)^2$	$[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[z_1, z_1] = x_0$	$L^{15}_{(1,3)}$	Nontrivial	R, C
(0, 4)	—	—	—	—	—
(5, 0)	—	$L = \langle x_0, y_0, z_0, v_0, w_0 \rangle$, $[x_0, y_0] = w_0$, $[z_0, v_0] = w_0$	$L^{16}_{(5,0)}$	Heisenberg Lie algebra	R, C
(5, 0)	—	$[x_0, y_0] = v_0$, $[x_0, z_0] = w_0$	$L^{17}_{(5,0)}$	Lie algebra	R, C
(5, 0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[y_0, w_0] = v_0$	$L^{18}_{(5,0)}$	Lie algebra	R, C
(5, 0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[y_0, z_0] = w_0$	$L^{19}_{(5,0)}$	Lie algebra	R, C
(5, 0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[x_0, v_0] = w_0$ equivalent to $[x_0, y_0] = w_0$, $[x_0, z_0] = y_0$, $[x_0, v_0] = z_0$	$L^{20}_{(5,0)}$	Lie algebra	R, C
(5, 0)	—	$[x_0, y_0] = z_0$, $[x_0, z_0] = v_0$, $[x_0, v_0] = w_0$, $[v_0, z_0] = W_0$	$L^{21}_{(5,0)}$	Lie algebra	R, C
(4, 1)	E^1	$L = \langle x_0, y_0, z_0, v_0 \rangle \oplus \langle x_1 \rangle$ $[v_0, y_0] = x_0$, $[v_0, z_0] = z_0$, $[x_1, x_1] = x_0$	$L^{22}_{(4,1)}$	Nontrivial	R, C
(4, 1)	E^2	$[x_0, y_0] = z_0$, $[v_0, y_0] = x_0$, $[x_1, x_1] = z_0$	$L^{23}_{(4,1)}$	Nontrivial	R, C

Table I. Continued

Type	L	Characterization	Relation	Comments	K
(3, 2)	E^3	$L = \langle x_0, y_0, Z_0\rangle \oplus \langle x_1, y_1\rangle$, $[x_0, y_0] = z_0$, $[x_0, y_1] = x_1$	$L^{24}_{(3,2)}$	Trivial	R, C
(3, 2)	$(E^4)^1$	$[x_0, y_0] = z_0$, $[x_1, x_1] = z_0$, $[y_1, y_1] = z_0$	$L^{25}_{(3,2)}$	Nontrivial Heisenberg Lie superalgebra	R, C
(3, 2)	$(E^4)^2$	$[x_0, y_0] = z_0$, $[x_1, x_1] = z_0$, $[y_1, y_1] = -z_0$	$L^{26}_{(3,2)}$	Nontrivial	R, C
(3, 2)	(E^3)	$[x_0, y_0] = z_0$, $[x_0, y_1] = x_1$, $[y_1, y_1] = z_0$	$L^{27}_{(3,2)}$	Nontrivial	R, C
(2, 3)	$(C + A)^1$	$L = \langle x_0, y_0\rangle \oplus \langle x_1, y_1, z_1\rangle$, $[x_0, y_1] = x_1$, $[y_1, y_1] = y_0$, $[z_1, z_1] = y_0$	$L^{28}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(C + A)^2$	$[x_0, y_1] = x_1$, $[y_1, y_1] = y_0$, $[z_1, z_1] = -y_0$	$L^{29}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^1$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = x_0 + y_0$	$L^{30}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^2$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = -(x_0 + y_0)$	$L^{31}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^3$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[z_1, z_1] = x_0 - y_0$	$L^{32}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^4$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = y_0$	$L^{33}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^5$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = x_0 + y_0$	$L^{34}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^6$	$[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, z_1] = x_0 - y_0$	$L^{35}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^7$	$[x_1, x_1] = x_0$, $[x_1, z_1] = y_0$, $[y_1, z_1] = x_0$	$L^{36}_{(2,3)}$	Nontrivial	R, C
(2, 3)	$(2A_{1,1} + 3A)^8$	$[x_1, y_1] = x_0$, $[y_1, z_1] = y_0$	$L^{37}_{(2,3)}$	Nontrivial	R, C

Table I. Continued

Type	L	Characterization	Relation	Comments	K
(1, 4)	E^5	$L = \langle x_0 \rangle \oplus \langle x_1, y_1, z_1, v_1 \rangle$, $[x_0, y_1] = x_1$, $[x_0, v_1] = z_1$	$L^{38}_{(1,4)}$	Trivial	R, C
(1, 4)	E^6	$[x_0, y_1] = x_1$, $[x_0, z_1] = y_1$, $[x_0, v_1] = z_1$	$L^{39}_{(1,4)}$	Trivial	R, C
(1, 4)	$(A_{1,1} + 4A)^1$	$[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[z_1, z_1] = x_0$, $[v_1, v_1] = x_0$	$L^{40}_{(1,4)}$	Nontrivial	R, C
(1, 4)	$(A_{1,1} + 4A)^2$	$[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[z_1, z_1] = x_0$, $[v_1, v_1] = -x_0$	$L^{41}_{(1,4)}$	Nontrivial	R
(1, 4)	$(A_{1,1} + 4A)^3$	$[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[z_1, z_1] = -x_0$, $[v_1, v_1] = -x_0$	$L^{42}_{(1,4)}$	Nontrivial	R

$$\begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & . & & & . \\ . & & . & . & 1 & . & & . \\ . & & & . & . & 0 & . & . \\ . & & & & . & . & 1 & 0 \\ 0 & & & & & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 & 0 \end{bmatrix}$$

Proof. Since the Lie superalgebra L is of type $(1, n)$, then

$$[L_0, L_0] = 0$$

$$[L_0, L_1] = [x_0, L_1] = ad_{x_0}(L_1)$$

$$[L_1, L_1] = \langle x_0 \rangle$$

The multiplication $[L_1, L_1]$ can be given by a symmetric bilinear form, say $B(x_1, y_1) = \alpha$, hence $[x_1, y_1] = \alpha x_0$. Assume that L_1 has the basis $B_{L_1} = \{x_1^1, x_1^2, \ldots, x_1^n\}$. Then we can choose B such that $B(x_1^i, x_1^j) = a_i \delta_{ij}$,

$$a_i \in \{0, 1, -1\} \text{ for } K = R, \; a_i \in \{0, 1\} \text{ for } K = C.$$

This means that $[x_1^i, x_1^i] = a_i x_0$ and $[x_1^i, x_1^j] = 0$. Suppose that $[L_0, L_1] \neq 0$. Then there exists an element x_1^k such that $[x_0, x_1^k] = \Sigma_{i=1}^n \gamma_i x_1^i \neq 0$.

Suppose also that $[L_1, L_1] \neq 0$. Hence there exists an element x_1^l such that $[x_1^l, x_1^l] = \pm x_0$. From the Jacobi identity we have

$$[[x_1^l, x_1^l], x_1^k] + [[x_1^l, x_1^k], x_1^l] + [[x_1^k, x_1^l], x_1^l] = 0$$

Hence $\Sigma_{i=1}^n \gamma_i x_1^i = 0$. This is a contradiction and we deduce that either $[L_0, L_1] = 0$ or $[L_1, L_1] = 0$.

(ii) Since $[L, L] \neq 0$, then we have $ad_{x_0} \neq 0$ or the symmetric bilinear form $B \neq 0$.

If $B \neq 0$, then we get $[L, [L, L]] = [L, x_0] = 0$ and L is nilpotent. If L is nilpotent of type $(1, n)$, then from (i) we deduce that $[x_1^i, x_1^i] = a_i x_0$.

If $ad_{x_0} \neq 0$, then ad_{x_0} is a nilpotent operator, since L is nilpotent. By using a basis transformation of L_1 and the fact that ad_{x_0} has only zero eigenvalue, one can put $ad_{x_0}(L_1)$ in the Jordan form.

If $ad_{x_0}(L_1)$ has the Jordan form and L is of type $(1, n)$, then one can see that L is a nilpotent Lie superalgebra by direct calculations.

2. SUPERVERSION OF KIRILLOV LEMMA

Let us consider a nilpotent Lie algebra G with one-dimensional center, in which Heisenberg Lie algebra H exists as a subalgebra with the same center $C(G) = C(H)$.

Lemma 2.1. Let G be a nilpotent Lie algebra with one-dimensional center, $\dim G \geq 3$. Then G is decomposable into the subspaces X, Y, Z, and W such that:

(i) $G = X \oplus Y \oplus Z \oplus W$.
(ii) $\dim X = \dim Y = \dim Z = 1$.
(iii) If $X = \langle x \rangle$, $Y = \langle y \rangle$, $Z = \langle z \rangle$, then

$$[x, y] = z, \qquad [y, w] = 0, \qquad \text{for all} \quad w \in W$$

Proof. Since G is a nilpotent Lie algebra, then there exists $n \in N$ such that $G^{(n)} = [G, G^{(n-1)}] = 0$, $G^{(n-1)} \neq 0$. It is clear that $G^{(n-1)} \subseteq C(G)$; then it follows that $G^{(n-1)} = C(G)$. We put $C(G) = Z = \langle z \rangle$. Let $y \in G^{(n-2)} \setminus G^{(n-1)}$, $y \neq 0$. We put $Y = \langle y \rangle$. Since G is nilpotent, there exists $x \in G$ with $[x, y] = z$. We put $X = \langle x \rangle$. It is clear that $X \cap Y = (0)$, $Y \cap Z = (0)$, and $X \cap Z = (0)$. We consider $\tilde{W} = \{g\colon g \in G, [y, g] = 0\}$. $\tilde{W}$ is a Lie subalgebra of G. It is clear that $Y \subseteq \tilde{W}$, $Z \subseteq \tilde{W}$. Thus $\tilde{W} = Y \oplus Z \oplus W$, where W is a certain vector subspace of G. Since $x \notin \tilde{W}$, then $X \cap \tilde{W} = (0)$ and we have

$$X \oplus Y \oplus Z \oplus W \subseteq G$$

Let $g \in G$, $[g, y] = \alpha.z$; then $[g - \alpha x, y] = 0$, so $g - \alpha x \in \tilde{W}$. Thus

$g = \alpha x + \tilde{w}$ and it follows that $G \subseteq X \oplus Y \oplus Z \oplus W$. Thus $G = X \oplus Y \oplus Z \oplus W$.

Remark. We have the following results:

(i) $X + Y + Z$

$$\cong \left\{ \begin{vmatrix} 0 & 0 & . & . & . & . & x & z \\ 0 & 0 & . & . & . & . & 0 & y \\ . & & & & & & & \\ . & & & & & & & \\ . & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}_{m.m} : x, y, z \in k, m \geq 3 \right\}$$

(ii) $G' = X \oplus Y \oplus Z$ is a Lie subalgebra of G with minimal dimension such that $C(G) \nsubseteq G'$ and $C(G) = C(G')$.

(iii) $G' = X \oplus Y \oplus Z$ is the minimal dimensional Lie subalgebra (dim $G' \geq 3$) with (1) $C(G) \nsubseteq G'$, $C(G) = \langle z \rangle$ and (2) there exist $x \in G'$, $y \in G'$ such that $[x, y] = z$.

Let $L = L_0 \oplus L_1$ be a nilpotent Lie superalgebra over R or C with one-dimensional center $= Kz$, where z is an even element. Then we have the following result.

Lemma 2.2. (i) L decomposes into the subspaces $X = \langle x \rangle$, $Y = \langle y \rangle$, $Z = \langle z \rangle$, and W.

(ii) x, y are homogeneous of the same parity, $[x, y] = z$ and $[y, w] = 0$.

(iii) $\tilde{W}$ is a superideal of codimension one, $\tilde{W} = Z \oplus Y \oplus W$, $L = \tilde{W} \oplus K\mathrm{x}$.

Proof. Let h/Z be the center of L/Z and let y be a homogeneous element of $h \backslash Z$ which does not belong to Z. For all $u \in L$, we have $[u, y] = \mu(u).z$, where $\mu \in L^*$ and L^* is the dual space of L. We have $\mu \neq 0$ since $y \notin Z$, hence we can find an element x which is homogeneous of the same parity as y such that $\mu(x) = 1$. Now the centralizer $\tilde{W}$ of Y in L, namely Ker μ, is of codimension one in L. Since L is nilpotent, $\tilde{W}$ is an ideal.

From remarks (ii) and (iii) we have the following questions.

1. What are the types of minimal dimensional Lie super subalgebra $L' \subseteq L$ with (i) $C(L) \nsubseteq L'$, (ii) $C(L) = C(L')$?

2. (i) What are the types of minimal dimensional Lie super subalgebra $L' \subseteq L$ with $C(L) \nsubseteq L'$, $C(L) = \langle c \rangle$? (ii) Do there exist $x \in L'$, $y \in L'$ such that $[x, y] = c$?

In the first question we must have $C(L) = C(L')$, but in the second question we want $C(L) \nsubseteq C(L')$

Unfortunately, we do not have a satisfactory answer for the first question. Let us assume that $C(L) = \langle c \rangle$ and dim $L' \leq 5$; then we have the following question. Let L be a nilpotent Lie superalgebra with dim $C(L) = 1$, $C(L) = \langle c \rangle$. (i) What are the types of minimal dimensional Lie super subalgebra $L' \subseteq L$ with $C(L) \nsubseteq L'$? (ii) Do there exist $x \in L'$, $y \in L'$ such that $[x, y] = c$? (iii) dim $L' \leq 5$?

Since L is nilpotent, so is L' and since L' is minimal dimensional, then L' can only be of the type $L^1_{(1,1)}$ to $L^{42}_{(1,4)}$ (see Table II). From (ii) it follows that $\langle c \rangle \subseteq [L', L']$; since $C(L)$ is a super subalgebra of L, then either $c \in L_0$ or $c \in L_1$, and we write $C(L) = \langle c_0 \rangle$ or $\langle c_1 \rangle$.

Remarks. 1. Type 6 exists only for $K = R$, types 1–5 exist for $K = R$ or C.

2. The type of the minimal Lie superalgebra L' is well determined (unique) except for type 6. Type 6 is connected with type 5; if we have a minimal Lie super subalgebra L' of type 6, we find a minimal Lie super subalgebra L' of type 5.

3. In types 1, 2, and 3 we have $C(L) = C(L')$ and types 2 and 3 regardless of the grading are isomorphic to the classical Heisenberg Lie algebra.

Proposition 2.3. Let L be a nilpotent Lie superalgebra over $K = R$ or C with dim $L \geq 2$, $C(L) = \langle c \rangle \neq (0)$. (i) There is a minimal dimensional Lie super subalgebra L' of L with $C(L) \subseteq L'$. (ii) There exist $x \in L'$, $y \in L'$ such that $[x, y] = c$ is one of the following types:

1. $L' = \langle c_0 \rangle \oplus \langle x_1 \rangle$, $[x_1, x_1] = c_0$.
2. $L' = \langle x_0, y_0, c_0 \rangle$, $[x_0, y_0] = c_0$.
3. $L' = \langle x_0 \rangle \oplus \langle x_1, c_1 \rangle$, $[x_0, x_1] = c_1$.
4. $L' = \langle x_0, y_0 \rangle \oplus \langle x_1, c_1 \rangle$, $[x_0, x_1] = c_1$, $[x_1, x_1] = y_0$.
5. $L' = \langle x_0, c_0 \rangle \oplus \langle x_1, y_1 \rangle$, $[x_1, y_1] = c_0$, $[x_1, x_1] = x_0$.
6. $L' = \langle x_0, c_0 \rangle \oplus \langle x_1, y_1 \rangle$, $[x_1, y_1] = c_0$, $[y_1, y_1] = x_0$, $[x_1, x_1] = x_0$.

Proof. Since L is nilpotent, then there exists $n \in N$ such that $L^{(1)} = L$, $L^{(2)} = [L, L^{(1)}], \ldots, L^{(n-1)} = [L, L^{(n-2)}] \neq (0)$, $L^{(n)} = (0)$. It is clear that $L^{(n-1)} \subseteq C(L)$ and since dim $C(L) = 1$, then $L^{(n-1)} = C(L) = \langle c \rangle$. Suppose first that $c = c_0 \in L_0$ and let $y \in (L^{(n-2)} L^{(n-1)}) \neq (0)$. We assume $y = y_0 \in L_0^{(n-2)}$. Since $y_0 \notin C(L)$, then $[L, y_0] \subseteq L^{(n-1)} = \langle c_0 \rangle$, and there exists $x_0 \in L_0$ with $[x_0, y_0] = c_0$. Thus $L = \langle x_0, y_0, c_0 \rangle$ is of type 2.

Let us now consider $y = y_1 \in L_1^{(n-2)}$. Since $y_1 \notin C(L)$, then $[L, y_1] \subseteq L^{(n-1)} = \langle c_0 \rangle$, and there exist $x_1 \in L_1$ with $[x_1, y_1] = c_0$. If $x_1 = \lambda y_1$ (we take $\lambda = 1$), then $L = \langle c_0 \rangle \oplus \langle x_1 \rangle$ is of type 1. If $x_1 \neq \lambda y_1$, then $[y_1, y_1] = \beta c_0$ ($\beta = 1$) and $L' = \langle c_0 \rangle \oplus \langle y_1 \rangle$ is of type 1.

Let us assume that $[y_1, y_1] = 0$, $[x_1, x_1] = 0$. Then it follows that $[1/2\, x_1 + y_1, 1/2\, x_1 + y_1] = c_0$ and $L' = \langle c_0 \rangle \oplus \langle 1/2\, x_1 + y_1 \rangle$ is of type 1.

Table II

L'	Relations	K	Type
$L^1_{(1,1)}$	$[L^1_{(1,1)}, L^1_{(1,1)}] = \langle x_0\rangle \to x_0 = \lambda c_0$; without loss of generality we take $\lambda = 1$, $L' = \langle c_0\rangle \oplus \langle x_1\rangle$, $[x_1, x_1] = c_0$	R, C	1
$L^2_{(3,0)}$	$[L^2_{(3,0)}, L^2_{(3,0)}] = \langle z_0\rangle$, $c_0 = z_0 \to L' = \langle x_0, y_0, z_0\rangle$, $[x_0, y_0] = c_0$	R, C	2
$L^3_{(1,2)}$	$c_0 = y_1 \to L' = \langle x_0\rangle \oplus \langle x_1, c_1\rangle$, $[x_0, x_1] = c_1$	R, C	3
$L^4_{(1,2)}$	$c_0 = x_0 \to L' = \langle c_0\rangle \oplus \langle x_1\rangle$ is a minimal subalgebra	R, C	—
$L^5_{(1,2)}$	Analogous to $L^4_{(1,2)}$	—	—
$L^6_{(4,0)}$	$c_0 = \alpha v_0 + \beta z_0 \to L'' = \langle x_0, \beta y_0 + \alpha z_0, c_0\rangle$ is a minimal algebra since $[x_0, \alpha z_0 + \beta y_0] = c_0$	R, C	—
$L^7_{(3,1)}$	$c_0 = z_0 \to L'' = \langle x_0, y_0, c_0\rangle$ is a minimal subalgebra ($L'' = \langle c_0\rangle \oplus \langle x_1\rangle$ is also a minimal subalgebra) R, C	—	
$L^8_{(2,2)}$	$[L^8_{(2,2)}, L^8_{(2,2)}] = \langle y_0\rangle \oplus \langle y_1\rangle$; if $y_0 = c_0$, then $L'' = \langle c_0\rangle \oplus \langle x_1\rangle$ is minimal; if $y_1 = c_1$, so is $L' = \langle x_0, y_0\rangle \oplus \langle x_1, c_1\rangle$ with $[x_0, x_1] = c_1$ $[x_1, x_1] = y_0$	R, C	4
$L^9_{(2,2)}$	$[L^9_{(2,2)}, L^9_{(2,2)}] = \langle x_0, y_0\rangle$; we use the equivalent form of $L^9_{(2,2)}$, i.e., $[x_1, x_1] = x_0$, $[y_1, y_1] =0$, $[x_1, y_1] = y_0$; if we take $c_0 = \alpha x_0 + \beta y_0$, $\alpha.\beta \neq 0 \to L'$ is not minimal dimensional; $L'' = \langle c_0\rangle \oplus \left(\sqrt{\alpha}\, x_1 + \frac{\beta}{2\sqrt{\alpha}}\, y_1\right)$ is minimal for $k = C$; $L'' = \langle c_0\rangle \oplus (\pm\sqrt{\lvert\alpha\rvert}\, x_1 + (\beta/2\sqrt{\lvert\alpha\rvert}) y_1)$ is minimal for $K = R$; if $x_0 = c_0 \to L'$ is also not minimal, let $y_0 = c_0 \to L' = \langle x_0, c_0\rangle \oplus \langle x_1, y_1\rangle$ with $[x_1, x_1] = x_0$, $[x_1, y_1] = c_0$	R, C	5
$L^{10}_{(2,2)}$	$[L^{10}_{(2,2)}, L^{10}_{(2,2)}] = \langle x_0, y_0\rangle$; $c_0 = \alpha x_0 + \beta y_0 \to L'$ is not minimal (if $\alpha = 0$ or $\beta = 0$, then L' is not minimal); for $K = C$, L' is not minimal; for $K = R$ and $\alpha.\beta < 0$, ($\alpha.\beta > 0 \to L'$ is not minimal), we take $\alpha > 0$, $\beta < 0$; then we denote $x'_1 = \sqrt{\alpha}\, x_1 + \sqrt{\lvert\beta\rvert}\, y_1$, $x'_0 = \alpha x_0 + \lvert\beta\rvert\, y_0$, $y'_1 = \sqrt{\alpha}\, x_1 - \sqrt{\lvert\beta\rvert}\, y_1$, $y'_0 = \alpha x_0 - \lvert\beta\rvert\, y_0 = \alpha x_0 + \beta y_0 = c_0$, $[x'_1, y'_1] = c_0$, $[x'_1, x'_1] = x'_0$, $[y'_1, y'_1] = x'_0$; thus $L' = \langle x_0, c_0\rangle \oplus \langle x_1, y_1\rangle$ with $[x_1, y_1] = c_0$, $[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$	R	6
$L^{11}_{(2,2)}$	$[L^{11}_{(2,2)}, L^{11}_{(2,2)}] = \langle x_0, y_0\rangle$; we use the equivalent form of $L^{11}_{(2,2)}$, i.e., $[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, $[x_1, y_1] = y_0$; $c_0 = \alpha x_0 + \beta y_0 \to L'$ is not minimal, for example, if $\beta = 0 \to L' = \langle c_0\rangle \oplus \langle x_1\rangle$ is minimal	R	—

Table II. Continued

L'	Relations	K	Type
$L^{12}_{(2,2)}$	Analogous to $L^{11}_{(2,2)}$		
$L^{13}_{(1,3)}$	$[L^{13}_{(1,3)}, L^{13}_{(1,3)}] = \langle y_1, z_1\rangle$; $c_1 = \alpha y_1 + \beta z_1 \to L'' = \langle x_0\rangle \oplus \langle \alpha x_1 + \beta y_1, c_1\rangle$ is minimal	R, C	—
$L^{14}_{(1,3)}$	Analogous to $L^{4}_{(1,2)}$	—	—
$L^{15}_{(1,3)}$	Analogous to $L^{4}_{(1,2)}$	—	—
$L^{16}_{(5,0)}-L^{21}_{(5,0)}$	Analogous to $L^{6}_{(4,0)}$	—	—
$L^{22}_{(4,1)}$	Analogous to $L^{7}_{(3,1)}$	—	—
$L^{23}_{(4,1)}$	Analogous to $L^{7}_{(3,1)}$	—	—
$L^{24}_{(3,2)}$	Analogous to $L^{8}_{(2,2)}$	—	—
$L^{25}_{(3,2)}$	Analogous to $L^{4}_{(1,2)}$	—	—
$L^{26}_{(3,2)}$	Analogous to $L^{4}_{(1,2)}$	—	—
$L^{27}_{(3,2)}$	Analogous to $L^{8}_{(2,2)}$	—	—
$L^{28}_{(2,3)}$	Analogous to $L^{8}_{(2,2)}$	—	—
$L^{29}_{(2,3)}$	Analogous to $L^{8}_{(2,2)}$	—	—
$L^{30}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{31}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{32}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{33}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{34}_{(2,3)}$	Analogous to $L^{11}_{(2,2)}$	—	—
$L^{35}_{(2,3)}$	Analogous to $L^{11}_{(2,2)}$	—	—
$L^{36}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{37}_{(2,3)}$	Analogous to $L^{10}_{(2,2)}$	—	—
$L^{38}_{(1,4)}$	Analogous to $L^{12}_{(1,3)}$	—	—
$L^{39}_{(1,4)}$	Analogous to $L^{12}_{(1,3)}$	—	—
$L^{40}_{(1,4)}$	Analogous to $L^{13}_{(1,3)}$	—	—
$L^{41}_{(1,4)}$	Analogous to $L^{13}_{(1,3)}$	—	—
$L^{42}_{(1,4)}$	Analogous to $L^{14}_{(1,3)}$	—	—

Suppose also that $[y_1, y_1] = 0$, $[x_1, x_1] = x_0 \neq 0$. Since $[[x_1, x_1], x_1] = 0$, so is $[x_0, x_1] = 0$. Take $x_0 = -\lambda c_0$ ($\lambda = 1$); then $L' = \langle c_0\rangle \oplus \langle x_1\rangle$ is of type 1. Now since $[x_0, y_1] \in L^{(n-1)} = \langle c_0\rangle$, we get $[x_0, y_1] = 0$ and $L' = \langle x_0, c_0\rangle \oplus \langle x_1, y_1\rangle$ is of type 5.

Second, consider $c = c_1 \in L_1$ and let $y \in (L^{(n-2)}L^{(n-1)}) \neq (0)$, $y = y_1 \in L_1^{(n-1)}$. Since $y_1 \notin C(L)$, then $[L, y_1] \subseteq L^{(n-1)} = \langle c_1\rangle$, and there exists $x_0 \in L_0$ with $[x_0, y_1] = c_1$. We have $[x_0, x_1] = 0$, $[y_1, y_1] = 0$, since $[y_1, y_1] \in L^{(n-1)} = \langle c_1\rangle$. Thus $L' = \langle x_0\rangle \oplus \langle y_1, c_1\rangle$ is of type 3.

Suppose that $y = y_0 \in L_0^{(n-2)}$. Since $y_0 \notin C(L)$, then $[L, y_0] \subseteq L^{(n-1)} = \langle c_1\rangle$ and there exists $x_1 \in L_{-1}$ with $[x_1, y_0] = c_1$. Now, $[x_1, x_1] = 0$, so we get $L' = \langle y_0\rangle \oplus \langle x_1, c_1\rangle$ which is of type 3. Assume that $[x_1, x_1] = x_0 \neq 0$. We have $[y_0, y_0] = 0$ and $x_0 \neq \lambda y_0$, since $[[x_1, x_1], x_1] = [x_0, x_1] = 0$. On the other hand, $[x_0, y_0] \in \langle c_0\rangle$, which implies that $[x_0, y_0] = 0$. Thus $L' = \langle x_0, y_0\rangle \oplus \langle x_1, c_1\rangle$ is of type 4.

Remarks. 1. All subalgebras L' are nonsimple, since the minimal dimensional simple Lie superalgebra is of dimension 5 [6].

2. We have seen that the nilpotent Lie superalgebra L has L' subalgebra of type 1, 2, 3, 4, and 5. The dimension of a minimal L' cannot exceed 4, and with this technique we cannot find a minimal L' of L!

3. If L' is minimal of type 6, we can find at most an L'' of type 4 or 5, but since in type 6 we have $c \in L_0$ and in type 4 we have $c \in L_1$, we deduce that L'' is of type 5. Moreover, if the minimum dimension is 2, then L' is of type 1, and if the minimum dimension is 3, then L' is of type 2 or 3, but c is homogeneous of different degree in types 2 and 3. Thus we have only one type 2 or 3. Analogous reasoning for dimensions 4 and 5 and gives only one of type 4 or 5.

4. Heisenberg Lie superalgebra is a nilpotent Lie superalgebra with one-dimensional center which has no L'.

APPENDIX

Here I outline the arguments used in obtaining the classification table. I take as an example the nilpotent Lie superalgebra of type (2,2) to represent the technical arguments which are used to obtain the classification table.

Choose $L = L_{(2,2)} = \langle x_0, y_0\rangle \oplus \langle x_1, y_1\rangle$. Since L is nilpotent, then L_0 is also a nilpotent Lie algebra of dimension 2. This implies that $[L_0, L_0] = 0$. The dimension of $[L_0, L_1]$ must be ≤ 1 otherwise $[L_0, L_1] = L_1$ and L is not nilpotent.

Take dim $[L_0, L_1] = 1$; without loss of generality $[L_0, L_1] = \langle y_1\rangle \rightarrow [x_0, x_1] = y_1$, $[y_0, x_1] = \alpha y_1$, $[x_0, y_1] = 0$, $[y_0, y_1] = 0$, otherwise $[L_0, [L_0, L_1]] = [L_0, L_1]$ and L cannot be nilpotent.

Without loss of generality we can take $\alpha = 0$; otherwise we take the basis $x_0' = x_0$ and $y_0' = \alpha x_0 - y_0$.

From the Jacobi identity we get

$$0 = [[x_0, x_1], y_1] + [[x_1, y_1], x_0] - [[y_1, x_0], x_1]$$

$$= [y_1, y_1] \Rightarrow [y_1, y_1] = 0$$

$$0 = [[x_0, x_1], x_1] + [[x_1, x_1], x_0] - [[x_1, x_0], x_1]$$

$$= 2\,[y_1, x_1] \Rightarrow [y_1, x_1] = 0$$

$$[[x_1, x_1], x_1] = [\alpha x_0 + \beta y_0, x_1] = \alpha y_1 \Rightarrow [x_1, x_1] = \beta y_0$$

If $\beta = 0$, we have L with $[x_0, x_1] = y_1$, which can be derived from $L^3_{(1,2)}$.

If $\beta = 1$, then the nilpotent Lie superalgebra $L = L_{(2,2)}$ is given by $[x_0, x_1] = y_1$, $[x_1, x_1] = y_0$. This is the nilpotent Lie superalgebra $L^8_{2(2,2)}$.

Assume now that

$$\dim\,[L_0, L_1] = 0 \Rightarrow [L, L] = [L_1, L_1] \subseteq L_0 \Rightarrow \dim[L_1, L_1] \in \{1, 2\}$$

Take $\dim[L_1, L_1] = 1 \Rightarrow [L_1, L_1] = \langle x_0 \rangle$. We have L with $[x_1, x_1] = x_0$, which can be derived from $L^1_{(1,1)}$ or L with $[x_1, x_1] = x_0$, $[y_1, y_1] = x_0$, which can be derived from $L^1_{(1,2)}$, or L with $[x_1, x_1] = x_0$, $[y_1, y_1] = -x_0$, which can be derived from $L^5_{(1,2)}$.

Take $\dim[L_1, L_1] = 2 \Rightarrow [L_1, L_1] = \langle x_0, y_0 \rangle = \langle [x_1, x_1], [y_1, y_1], [x_1, y_1] \rangle$ and we continue the technical arguments to get $L^9_{(2,2)}$, $L^{10}_{(2,2)}$, $L^{11}_{(2,2)}$, and $L^{12}_{(2,2)}$. The final results are given as follows:

Lemma. Let L be a nilpotent Lie superalgebra of type (2,2) with the basis $\langle x_0, y_0, x_1, y_1 \rangle$ and $[x_1, x_1] = x_0$, $[y_1, y_1] = y_0$, $[x_1, y_1] = \alpha x_0 + \beta y_0$; then for $K = R$

$$L = L^9_{(2,2)} \qquad \text{for } \alpha.\beta = \frac{1}{4}$$

$$L = L^{10}_{(2,2)} \qquad \text{for } \alpha.\beta < \frac{1}{4}$$

$$L = L^{11}_{(2,2)} \qquad \text{for } \alpha.\beta > \frac{1}{4}$$

and for $K = C$

$$L = L^9_{(2,2)} \qquad \text{for } \alpha.\beta = \frac{1}{4}$$

$$L = L^{10}_{(2,2)} \qquad \text{for } \alpha.\beta \neq \frac{1}{4}$$

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